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A HIGHER ORDER THEORY FOR FREE VIBRATION ANALYSIS OF CIRCULAR CYLINDRICAL SHELLS

A. BHIMARADDI
Department of Civil Engineering, University of Melbourne, Parkville, Victoria 3052, Australia

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Abstract—The free, undamped vibration of an isotropic circular cylindrical shell is analysed with higher order displacement model, giving rise to a more realistic parabolic variation of transverse shear strains. The method accounts for in-plane inertia, rotary inertia, and shear deformation effects on the dynamic response of cylindrical shells. The frequencies obtained from the present analysis, shear deformation theory, and Flügge theory are compared with the exact elasticity results. It is found that the frequencies predicted by the present analysis are closer to exact values than those predicted by the shear deformation theory, especially for cases with shorter wavelengths.

NOTATION

a, h, L radius, thickness and length of shell
x, θ, z, t axial, circumferential, radial and time coordinates
u, v, w axial, circumferential and radial displacements
u_0, v_0 axial and circumferential displacements at middle surface of shell
u_1, v_1 rotations in x and θ planes
m, n axial and circumferential mode numbers
E, G, μ, ρ Young's modulus, shear modulus, Poisson's ratio and mass density of the material
q_{zo}, q_{zi} applied forces in z direction on outer and inner surfaces of the shell
$\epsilon_x, \epsilon_\theta, \epsilon_z$ normal strains
$\gamma_{x\theta}, \gamma_{z\theta}, \gamma_{xz}$ shear strains
$\sigma_x, \sigma_\theta, \sigma_z$ normal stresses
$\tau_{x\theta}, \tau_{z\theta}, \tau_{xz}$ shear stresses
$N_x, N_\theta, N_{x\theta}, N_{\theta x}$ membrane forces per unit length of shell
$M_x, M_\theta, M_{x\theta}, M_{\theta x}$ bending and twisting moments per unit length of shell
Q_x, Q_θ transverse shear forces per unit length of shell

$$\Omega_0 = \frac{\pi}{h}\sqrt{\frac{G}{\rho}} \quad \mu_1 = \frac{1-\mu}{2} \quad \mu_2 = \frac{1+\mu}{2} \quad \lambda = \frac{m\pi a}{L}$$

$$B = \frac{Eh}{1-\mu^2} \quad D = \frac{Eh^3}{12(1-\mu^2)}$$

1. INTRODUCTION

The theory of shells is one of the most important applied branches of the theory of elasticity. Thin shell-type constructions are finding applications in the most diverse branches of technology. Apart from linearity between stress and strain, and deformations being considered small, classical thin shell theories are based on Kirchhoff–Love hypothesis; the implication is that thickness shear deformations are negligible.

The reliable prediction of the small deflection response characteristics of moderately thick shells require the use of shear deformation theories. A number of research workers [1–5] have studied the effects of shear deformation in the theory of vibration of homogeneous isotropic cylindrical shells. Their calculated results, as well as those of others, are summarized in the monograph of Leissa [6]. These studies confirmed that the effects of shear deformation can become quite significant for small radius-to-thickness or length-to-thickness ratios, as well as for shorter wavelengths of longer shells.

Although the above mentioned shear deformation theories yield acceptable solutions, they do not accurately predict response characteristics of the shell in cases such as: when accurate higher order frequencies are required; when accurate prediction of stress singularities is required; and when thermal gradient through the thickness of the shell becomes noticeable. Such problems can be addressed only when the three-dimensional elasticity methods are employed. But an exact three-dimensional analysis with general edge conditions is impractical. Hence it is necessary to formulate some approximate analysis which is sufficiently accurate and is practical.

Usually, in two-dimensional plate and shell theories, displacement components have been considered as power series expressions in normal coordinate (z). Various such formulations can be found in [7–10]. It may be said here that depending on the number of terms retained, in the power series expressions for displacement components, the number of unknown displacement parameters to be determined varies; and hence the accuracy and complexity of the problem also varies. Further discussions on the applicability and accuracy of such formulations, in the analysis of plates, can be found in [11].

In this paper an attempt is made to propose a two-dimensional theory for the analysis of isotropic homogeneous shells by adapting the displacement fields suggested by Kovařík [12] for orthotropic sandwich-type shells. Thus, the proposed method considers a total of five-unknowns, which are the mid-surface displacement quantities, and still maintains the higher order (cubic) polynomial form for in-plane displacement expressions. At the same time, a more realistic parabolic variation for transverse shear strains, with zero values at the extreme fibers, is achieved. Also, unlike some of the shear deformation theories, for example as in the case of [5, 8], the present analysis does not involve the determination of any unknown shear coefficients.

The primary intended application for the present higher order theory is in the field of laminated composite shells. However, the purpose of this paper is to evaluate the proposed method by comparing the results obtained from it, as well as from shear deformation theory [5], with the exact three-dimensional elasticity results for isotropic cylindrical shells [15]. The generalization of the present analysis to orthotropic homogeneous and laminated composite shells will be dealt with in a future publication.

The present analysis is based on the assumptions such as, small deflections, linear elasticity and isotropicity. Also, the logarithmic terms appearing in some of the force-displacement relations and strain energy expression are expanded in powers of h/a. Terms of order h^3/a^3 only are retained, neglecting higher order terms. This limits the application of the present theory for shells with higher h/a values, as is the case with most of the shell theories.

2. DERIVATION OF GOVERNING EQUATIONS OF MOTION

The components of displacements are assumed as follows (refer to the list of nomenclature):

$$u = u_0 + \xi u_1 - z w_{,x}$$

$$v = \left(\frac{a+z}{a}\right)\left(v_0 + \xi v_1 - \frac{z}{a+z} w_{,\theta}\right) \tag{1}$$

$$w = w$$

where $\xi = z[1 - (4z^2/3h^2)]$ and the comma indicates differentiation with respect to the letter(s) followed by it. The selection of the second term in u and v displacement expressions results in the parabolic variation of transverse shear strains as is evident from the eqns (3) below. The assumption of constant value for radial displacement (w), across the thickness of shell, facilitates to have zero transverse shear strains at the extreme fibers of the shell and also limits the complexity of the problem to a reasonable degree. Any additional terms in w expression would violate this situation. Thus, to have higher order expression for w we have to add appropriate terms to u and v expressions, to maintain zero values for transverse shear strains at the extreme fibers, which further complicates the problem. The strain-displacement relations of the three-dimensional elasticity [13] are:

$$\epsilon_x = u_{,x} \quad \epsilon_\theta = \frac{1}{a+z}(v_{,\theta} + w) \quad \epsilon_z = w_{,z}$$
$$\gamma_{x\theta} = v_{,x} + \frac{1}{a+z}u_{,\theta} \quad \gamma_{z\theta} = \frac{1}{a+z}w_{,\theta} + v_{,z} - \frac{v}{a+z} \quad \gamma_{xz} = w_{,x} + u_{,z}. \tag{2}$$

Substitution of eqns (1) into eqns (2) leads to the following strain-displacement relations

$$\epsilon_x = u_{0,x} + \zeta u_{1,x} - zw_{,xx} \qquad \epsilon_z = 0$$
$$\epsilon_\theta = \frac{1}{a}v_{0,\theta} + \frac{\zeta}{a}v_{1,\theta} - \frac{z}{a(a+z)}w_{,\theta\theta} + \frac{w}{a+z}$$
$$\gamma_{xz} = \left(1 - \frac{4z^2}{h^2}\right)u_1 \qquad \gamma_{z\theta} = \left(\frac{a+z}{a}\right)\left(1 - \frac{4z^2}{h^2}\right)v_1 \tag{3}$$
$$\gamma_{x\theta} = \left(\frac{a+z}{a}\right)v_{0,x} + \zeta\left(\frac{a+z}{a}\right)v_{1,x} + \left(\frac{1}{a+z}\right)u_{0,\theta} + \left(\frac{\zeta}{a+z}\right)u_{1,\theta} - \frac{z}{a}\left(\frac{2a+z}{a+z}\right)w_{,x\theta}.$$

Assuming $\sigma_z = 0$, the stress–strain relations can be written as

$$\sigma_x = \frac{E}{1-\mu^2}(\epsilon_x + \mu\epsilon_\theta) \quad \sigma_\theta = \frac{E}{1-\mu^2}(\epsilon_\theta + \mu\epsilon_x) \quad \tau_{x\theta} = G\gamma_{x\theta} \quad \tau_{z\theta} = G\gamma_{z\theta} \quad \tau_{xz} = G\gamma_{xz}. \tag{4}$$

We have the following definitions for stress resultants

$$(N_x, N_{x\theta}, M_x, M_{x\theta}, Q_x) = \int_{-h/2}^{h/2} (\sigma_x, \tau_{x\theta}, -z\sigma_x, -z\tau_{x\theta}, \tau_{xz})\left(\frac{a+z}{a}\right)\mathrm{d}z$$
$$(N_\theta, N_{\theta x}, M_\theta, M_{\theta x}, Q_\theta) = \int_{-h/2}^{h/2} (\sigma_\theta, \tau_{x\theta}, -z\sigma_\theta, -z\tau_{x\theta}, \tau_{\theta z})\,\mathrm{d}z. \tag{5}$$

In addition to the above we define the following

$$(M'_x, M'_\theta) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_\theta)\zeta\left(\frac{a+z}{a}\right)\mathrm{d}z$$
$$(M'_{x\theta}, M'_{\theta x}) = \int_{-h/2}^{h/2} \left(\frac{a+z}{a}, 1\right)\tau_{x\theta}\zeta\,\mathrm{d}z \tag{6}$$
$$N'_\theta = \int_{-h/2}^{h/2} \sigma_\theta\left(\frac{a+z}{a}\right)\mathrm{d}z.$$

Also,

$$\bar{M}_{\theta x} = \bar{M}_{x\theta} = M_{x\theta} + M_{\theta x}$$
$$Q'_x = M_{x,x} + \frac{1}{a}\bar{M}_{x\theta,\theta} + \frac{\rho h^3}{12}\left(\frac{1}{a}u_{0,tt} + \frac{12}{15}u_{1,tt} - w_{,xtt}\right)$$
$$Q'_\theta = \frac{1}{a}M_{\theta,\theta} + \bar{M}_{\theta x,x} + \frac{\rho h^3}{12}\left(\frac{2}{a}v_{0,tt} + \frac{12}{15}\left(1 + \frac{h^2}{7a^2}\right)v_{1,tt} - \frac{1}{a}w_{,\theta tt}\right).$$

The expressions for stress resultants are given in Appendix 1. Thus, strain energy of the shell under consideration is given as

$$U=\frac{1}{2}\left(\frac{E}{1-\mu^2}\right)\int_\theta\int_x\int_z[\epsilon_x{}^2+\epsilon_\theta{}^2+2\mu\epsilon_x\epsilon_\theta+\mu_1(\gamma_{x\theta}^2+\gamma_{xz}^2+\gamma_{z\theta}^2)]\left(\frac{a+z}{a}\right)a\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\theta. \quad (7)$$

The work done by the applied surface tractions, considering applied loads in z-direction only, may be written as

$$W=\frac{1}{2}\int_\theta\int_x 2(q_{zo}+q_{zi})wa\,\mathrm{d}x\,\mathrm{d}\theta. \quad (8)$$

The kinetic energy of the shell is given as

$$K=\frac{1}{2}\int_\theta\int_x\int_z\rho(u_{,t}^2+v_{,t}^2+w_{,t}^2)\left(\frac{a+z}{a}\right)a\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\theta. \quad (9)$$

Thus, the total potential energy of the shell is written as

$$\Pi=K+W-U. \quad (10)$$

Using Hamilton's principle [14] the following equations of equilibrium can be obtained

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15}\\ & L_{22} & L_{23} & L_{24} & L_{25}\\ & & L_{33} & L_{34} & L_{35}\\ & \text{Sym.} & & L_{44} & L_{45}\\ & & & & L_{55}\end{bmatrix}\begin{bmatrix}u_0\\u_1\\v_0\\v_1\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\a^2(q_{zo}+q_{zi})\end{bmatrix}. \quad (11)$$

Finally, the boundary conditions along the edge of shell require that, six linearly independent combinations of the following twelve boundary variables must be specified.

N_xu_0; M'_xu_1; $N_{x\theta}v_0$; $M'_{x\theta}v_1$; Q'_xw; $M_xw_{,x}$ (along $x=$ const.)

$N_{\theta x}u_0$; $M'_{\theta x}u_1$; $N'_\theta v_0$; $M'_\theta v_1$; $Q'_\theta w$; $M_\theta w_{,\theta}$ (along $\theta=$ const.).

The elements of L_{ij} matrix in eqns (11) are listed below.

$$L_{11}=B(a^2(\,)_{,xx}+\mu_1(\,)_{,\theta\theta})+\frac{D\mu_1}{a^2}(\,)_{,\theta\theta}+h\rho a^2(\,)_{,tt}$$

$$L_{12}=D\left(\frac{12a}{15}(\,)_{,xx}-\frac{\mu_1}{a}(\,)_{,\theta\theta}\right)+\frac{h^3\rho a}{15}(\,)_{,tt}$$

$$L_{13}=Ba\mu_2(\,)_{,x\theta}$$

$$L_{14}=\frac{4}{5}D\mu_2(\,)_{,x\theta}$$

$$L_{15}=B\mu a(\,)_{,x}+D\left(\frac{\mu_1}{a}(\,)_{,x\theta\theta}-a(\,)_{,xxx}\right)-\frac{h^3\rho}{12}(\,)_{,xtt}$$

$$L_{22}=-\frac{8}{15}B\mu_1a^2+D\left(\frac{204}{315}a^2(\,)_{,xx}+\mu_1(\,)_{,\theta\theta}\right)+\frac{17}{315}h^3a^2\rho(\,)_{,tt}$$

$$L_{23} = \frac{4}{5} D\mu_2(\,)_{,x\theta}$$

$$L_{24} = \frac{204}{315} D\mu_2 a(\,)_{,x\theta}$$

$$L_{25} = -D\left(\frac{12}{15}a^2(\,)_{,xxx} + \left(\frac{5-\mu}{9}\right)(\,)_{,x\theta\theta}\right) - \frac{h^3a^2\rho}{15}(\,)_{,xtt}$$

$$L_{33} = B[(\,)_{,\theta\theta} + \mu_1 a^2(\,)_{,xx}] + 3D\mu_1(\,)_{,xx} + ha^2\rho(\,)_{,tt} + \frac{h^3\rho}{2}(\,)_{,tt}$$

$$L_{34} = D\left[\frac{12}{5}\left(a + \frac{h^2}{21a}\right)\mu_1(\,)_{,xx} + \frac{12}{15a}(\,)_{,\theta\theta}\right] + \frac{2}{15}h^3\rho a(\,)_{,tt}$$

$$L_{35} = B(\,)_{,\theta} - D\left(\frac{3-\mu}{2}\right)(\,)_{,xx\theta} - \frac{h^3\rho}{6}(\,)_{,\theta tt}$$

$$L_{44} = -\frac{8}{15}B\mu_1 a^2 + D\left[\frac{12}{315}\left(17a^2 + \frac{83}{12}h^2\right)\mu_1(\,)_{,xx} + \frac{204}{315}(\,)_{,\theta\theta} - \frac{24}{35}\mu_1\right] + \frac{h^3\rho}{315}\left(17a^2 + \frac{83}{12}h^2\right)(\,)_{,tt}$$

$$L_{45} = D\left[-\frac{12}{15a}(\,)_{,\theta\theta\theta} + \frac{12}{15}\left(a + \mu_1\frac{h^2}{7a}\right)(\,)_{,xx\theta}\right] - \frac{h^3a\rho}{15}\left(1 + \frac{h^2}{7a^2}\right)(\,)_{,\theta tt}$$

$$L_{55} = B + D\left[a^2(\,)_{,xxxx} + 2(\,)_{,xx\theta\theta} + \frac{1}{a^2}(\,)_{,\theta\theta\theta\theta} + \frac{2}{a^2}(\,)_{,\theta\theta} + \frac{1}{a^2}\right] - h\rho a^2(\,)_{,tt} + \frac{h^3\rho}{12}[a^2(\,)_{,xxtt} + (\,)_{,\theta\theta tt}].$$

In deriving the equilibrium eqns (11) and stress resultants (eqns 5 and 6) logarithmic terms are expanded in powers of h/a and terms of order h^3/a^3 only are retained. It may be observed that Flügge's equations of motion can be obtained from eqns (11) by deleting rows and columns (2nd and 4th) corresponding to u_1 and v_1 displacements from L_{ij} matrix and corresponding rows on right hand side load vector.

3. RESULTS AND DISCUSSIONS

Free vibration of simply supported shell with no axial constraint is analysed using equations of motion (11). The solution of these equations is assumed in the following modal form,

$$u_0 = A_1\Phi_1(x,\theta)\,\mathrm{Sin}\,\Omega t; \quad u_1 = A_2\Phi_1(x,\theta)\,\mathrm{Sin}\,\Omega t$$
$$v_0 = A_3\Phi_2(x,\theta)\,\mathrm{Sin}\,\Omega t; \quad v_1 = A_4\Phi_2(x,\theta)\,\mathrm{Sin}\,\Omega t \tag{12}$$
$$w = A_5\Phi_3(x,\theta)\,\mathrm{Sin}\,\Omega t$$

where,

$$\Phi_1(x,\theta) = \mathrm{Cos}\,(m\pi x/L)\,\mathrm{Cos}\,n\theta$$
$$\Phi_2(x,\theta) = \mathrm{Sin}\,(m\pi x/L)\,\mathrm{Sin}\,n\theta$$
$$\Phi_3(x,\theta) = \mathrm{Sin}\,(m\pi x/L)\,\mathrm{Cos}\,n\theta.$$

Substituting eqns (12) in eqns (11), in the absence of applied forces, the problem reduces to finding the eignevalues and eignvectors of the following equation

$$Xp = \Phi^2 Yp \tag{13}$$

where X, Y are 5×5 matrices and $p^T = (A_1, A_2, A_3, A_4, A_5)$. The following shell geometric and material parameters are taken for comparing of results with exact elasticity analysis [15].

$$\lambda = 0.5\pi,\ \pi,\ 2\pi,\ 4\pi; \quad \mu = 0.3$$

$$h/a = 0.06,\ 0.1,\ 0.12,\ 0.18$$

$$n = 1,\ 2,\ 3,\ 4.$$

The results are presented in the tabular form along with the other theories. The values of the shear correction factors, used in calculating the numerical results for the shear deformation theory, have been taken as $\pi^2/12$. It may be said here that, for each combination of m and n values, Flügge theory yields 3-frequencies whereas the present analysis and the shear deformation theory yield 5-frequencies. The two extra frequencies correspond to thickness shear modes.

It was observed, for various parameters considered, that the frequency values, except those corresponding to flexural mode of vibration, were in good agreement with elasticity results. Hence the lowest natural frequencies are only considered for comparing the results obtained from various theories. These values are tabulated in Tables 1 and 2 for different parameters considered.

From these tables one may observe that Flügge theory over-estimates the frequencies for all parameters considered. The error in frequency values obtained from Flügge theory

Table 1. Comparison of lowest natural frequency parameters (Ω/Ω_0)

		h/a=0.06				h/a=0.10			
λ		n=1	n=2	n=3	n=4	n=1	n=2	n=3	n=4
	E	0.01853	0.01089	0.00826	0.01010	0.03100	0.01907	0.01814	0.02615
	P	0.01853	0.01090	0.00828	0.01011	0.03101	0.01911	0.01819	0.02618
0.5π	S	0.01853	0.01090	0.00828	0.01011	0.03101	0.01910	0.01819	0.02617
	F	0.01853	0.01090	0.00831	0.01019	0.03101	0.01913	0.01838	0.02679
	E	0.02781	0.02213	0.01816	0.01745	0.04784	0.03927	0.03643	0.04046
	P	0.02781	0.02214	0.01818	0.01748	0.04785	0.03978	0.03651	0.04051
π	S	0.02781	0.02214	0.01818	0.01748	0.04785	0.03977	0.03650	0.04049
	F	0.02782	0.02215	0.01823	0.01761	0.04791	0.03994	0.03697	0.04161
	E	0.03691	0.03611	0.03565	0.03631	0.07618	0.07684	0.07935	0.08475
	P	0.03692	0.03612	0.03566	0.03632	0.07615	0.07682	0.07931	0.08467
2π	S	0.03692	0.03612	0.03566	0.03630	0.07612	0.07677	0.07924	0.08457
	F	0.03717	0.03644	0.03610	0.03695	0.07842	0.07954	0.08287	0.08950
	E	0.08639	0.08748	0.08933	0.09199	0.20529	0.20802	0.21261	0.21906
	P	0.08639	0.08728	0.08911	0.09175	0.20478	0.20678	0.21132	0.21771
4π	S	0.08611	0.08718	0.08902	0.09165	0.20360	0.20628	0.21077	0.21710
	F	0.09161	0.09290	0.09510	0.09824	0.23623	0.23995	0.24620	0.25502

E- Elasticity [15], P- Present, S- Shear Deformation [5], F- Flugge [16]

Table 2. Comparison of lowest natural frequency parameters (Ω/Ω_0)

		h/a=0.12				h/a=0.18			
λ		n=1	n=2	n=3	n=4	n=1	n=2	n=3	n=4
	E	0.03730	0.02359	0.02462	0.03686	0.05652	0.03929	0.04996	0.07821
	P	0.03730	0.02365	0.02470	0.03687	0.05653	0.03944	0.05009	0.07833
0.5π	S	0.03730	0.02365	0.02469	0.03684	0.05653	0.03944	0.05002	0.07797
	F	0.03731	0.02371	0.02512	0.03813	0.05656	0.03985	0.05219	0.08387
	E	0.05853	0.04978	0.04789	0.05545	0.09402	0.08545	0.09093	0.11205
	P	0.05856	0.04986	0.04799	0.05550	0.09409	0.08562	0.09109	0.11202
π	S	0.05856	0.04985	0.04796	0.04796	0.09407	0.08555	0.09091	0.11170
	F	0.05872	0.05024	0.04899	0.05775	0.09509	0.08782	0.09616	0.12215
	E	0.10057	0.10234	0.10688	0.11528	0.18894	0.19467	0.20616	0.22450
	P	0.10047	0.10224	0.10674	0.11508	0.18832	0.19403	0.20544	0.22361
2π	S	0.10040	0.10214	0.10661	0.11488	0.18800	0.19358	0.20478	0.22269
	F	0.10517	0.10781	0.11387	0.12454	0.20923	0.21805	0.23486	0.26094
	E	0.27491	0.27849	0.28447	0.29287	0.50338	0.50937	0.51934	0.53325
	P	0.27286	0.27641	0.28233	0.29064	0.49818	0.50418	0.51416	0.52808
4π	S	0.27197	0.27547	0.28131	0.28951	0.49479	0.50058	0.51021	0.52366
	F	0.32960	0.33479	0.34349	0.35571	0.67100	0.68056	0.69634	0.71803

E- Elasticity [15], P- Present, S- Shear Deformation [5], F- Flügge [16]

increases for increased values of λ, n, and h/a. For example, the error is about $+16\%$ for $\lambda = 4\pi$, $n = 4$, $h/a = 0.1$ and $+35\%$ for $\lambda = 4\pi$, $n = 4$, $h/a = 0.18$.

The present analysis and the shear deformation theory over-estimate the frequencies for lower values of $\lambda(<\pi)$, and under-estimate the same for higher values of $\lambda(>\pi)$. For lower values of $\lambda(<\pi)$, the maximum error in the present analysis and in shear deformation theory is about $+0.4\%$. For higher values of $\lambda(>\pi)$, the maximum error in the present analysis is about -1%, and in the shear deformation theory it is about -1.8%; for $\lambda = 4\pi$, $n = 4$, $h/a = 0.18$.

It may be observed that the frequency values predicted by Flügge theory are bound from above. Whereas, better approximations such as, the shear deformation theory and the present analysis, bound eigenvalues from above for longer axial wavelengths ($\lambda < \pi$), and from below for shorter axial wavelengths ($\lambda > \pi$). Also, it may be seen that the bound becomes closer, in almost all cases, as eigenvalues converge towards the exact solution, as is evident in the case of present analysis.

4. CONCLUSIONS

A higher order theory for undamped dynamic response of an isotropic circular cylindrical shell is developed. The method accounts for in-plane inertia, rotary inertia and shear deformation effects. The proposed method assumes parabolic variation, across the thickness of shell, for transverse shear strains. From numerical results presented, for various shell geometric parameters, it may be concluded that the frequencies predicted by the present analysis are closer to the exact values, than those predicted by the shear deformation theory; especially for the cases with shorter wavelengths.

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APPENDIX 1

Force-displacement relations

$$N_x = B\left(u_{0,x} + \frac{\mu}{a}v_{0,\theta} + \frac{\mu}{a}w\right) + D\left(\frac{12}{15a}u_{1,x} + \frac{12h^2}{15a^2}v_{1,\theta} - \frac{1}{a}w_{,xx}\right)$$

$$N_\theta = B\left(\frac{1}{a}v_{0,\theta} + \mu u_{0,x} + \frac{w}{a}\right) + \frac{D}{a^3}(w_{,\theta\theta} + w)$$

$$N_{x\theta} = B\mu_1\left(v_{0,x} + \frac{1}{a}u_{0,\theta}\right) + \frac{D}{a}\mu_1\left(\frac{1}{a}v_{0,x} + \frac{24}{15}v_{1,x} - \frac{1}{a}w_{,x\theta}\right)$$

$$N_{\theta x} = B\mu_1\left(v_{0,x} + \frac{1}{a}u_{0,\theta}\right) + \frac{D}{a}\mu_1\left(\frac{1}{a^2}u_{0,\theta} - \frac{1}{a}u_{1,\theta} + \frac{12}{15}v_{1,x} + \frac{1}{a}w_{,x\theta}\right)$$

$$M_x = -\frac{D}{a}\left(u_{0,x} + \frac{12a}{15}u_{1,x} + \frac{\mu}{a}v_{0,\theta} + \frac{12\mu}{15}v_{1,\theta} - aw_{,xx} - \frac{\mu}{a}w_{,\theta\theta}\right)$$

$$M_\theta = -\frac{D}{a}\left(\frac{12}{15}\mu a u_{1,x} + \frac{12}{15}v_{1,\theta} - \mu a w_{,xx} - \frac{w}{a} - \frac{1}{a}w_{,\theta\theta}\right)$$

$$M_{x\theta} = -\frac{D}{a}\mu_1\left[2v_{0,x} + \frac{12}{15}\left(a + \frac{h^2}{7a}\right)v_{1,x} + \frac{12}{15}u_{1,\theta} - 2w_{,x\theta}\right]$$

$$M_{\theta x} = -\frac{D}{a}\mu_1\left(v_{0,x} - \frac{1}{a}u_{0,\theta} + \frac{12}{15}av_{1,x} + u_{1,\theta} - 2w_{,x\theta}\right)$$

$$Q_x = \frac{2}{3}B\mu_1 u_1 \qquad Q_\theta = \frac{2}{3}B\mu_1 v_1$$

$$N'_\theta = B\left(\mu u_{0,x} + \frac{1}{a}v_{0,\theta} + \frac{w}{a}\right) + \frac{D}{a}\left(\frac{12}{15}\mu u_{1,x} + \frac{12}{15a}v_{1,\theta} - \mu w_{,xx}\right)$$

$$M'_x = \frac{12D}{15a}\left(u_{0,x} + \frac{17}{21}au_{1,x} + \frac{\mu}{a}v_{0,\theta} + \frac{17}{21}\mu v_{1,\theta} - aw_{,xx} - \frac{\mu}{a}w_{,\theta\theta}\right)$$

$$M'_\theta = \frac{12D}{15a}\left(\mu u_{0,x} + \frac{17}{21}a\mu u_{1,x} + \frac{1}{a}v_{0,\theta} + \frac{17}{21}v_{1,\theta} - \mu a w_{,xx} - \frac{\mu}{a}w_{,\theta\theta}\right)$$

$$M'_{x\theta} = \frac{12D\mu_1}{15a}\left[\frac{1}{a}u_{0,\theta} + \frac{17}{21}u_{1,\theta} + \left(\frac{17a}{21} + \frac{83h^2}{252a}\right)v_{1,x} + \left(3 + \frac{h^2}{7a^2}\right)v_{0,x} - \left(2 + \frac{h^2}{7a^2}\right)w_{,x\theta}\right]$$

$$M'_{\theta x} = \frac{D\mu_1}{a}\left(-\frac{1}{a}u_{0,\theta} + u_{1,\theta} + \frac{12}{15}v_{0,x} + \frac{204}{315}av_{1,x} - 2w_{,x\theta}\right).$$